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CHARACTERIZATIONS OF PARETO, WEIBULL AND POWER FUNCTION DISTRIBUTIONS BASED ON GENERALIZED ORDER STATISTICS

Mohammad Ahsanullah* and G.G. Hamedani**

Abstract. Characterizations of probability distributions by different regression conditions on generalized order statistics has attracted the attention of many researchers. We present here, characterization of Pareto and Weibull distributions based on the conditional expectation of generalized order statistics extending the characterization results reported by Jin and Lee (2014). We also present a characterization of the power function distribution based on the conditional expectation of lower generalized order statistics.

1. Introduction

The concept of generalized order statistics (gos) was introduced by Kamps [8] in terms of their joint *pdf* (probability density function). The order statistics, record values, k-record values, Pfifer records and progressive type II order statistics are special cases of the gos. The $rv's$ (random variables) $X(1,n,m,k)$, $X(2,n,m,k)$, ..., $X(n,n,m,k)$, $k>0$, $m\in\mathbb{R}$, are n gos from an absolutely continuous *cdf* (cumulative distribution function) F with corresponding *pdf* f if their joint *pdf*, $f_{1,2,\dots,n}(x_1,x_2,\dots,x_n)$, can be written as

(1.1)
$$f_{1,2,\dots,n}(x_1,x_2,\dots,x_n)=k\left(\Pi_{j=1}^{n-1}\gamma_j\right)\left[\Pi_{j=1}^{n-1}\left(\overline{F}(x_j)\right)^m f(x_j)\right]\times$$
$$\left(\overline{F}(x_n)\right)^{k-1}f(x_n),\ F^{-1}(0+)<x_1<x_2<\dots<x_n<F^{-1}(1-),$$

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Correspondence should be addressed to G.G. Hamedani, g.hamedani@mu.edu.

where $\overline{F}(x) = 1 - F(x)$ and $\gamma_j = k + (n-j)(m+1)$ for all j, $1 \leq j \leq n$, k is a positive integer and $m \geq -1$.

If $k = 1$ and $m = 0$, then $X(r,n,m,k)$ reduces to the ordinary rth order statistic and (1.1) will be the joint *pdf* of order statistics $X_{1,n} \leq X_{2,n} \leq ... \leq X_{n,n}$ from F. If $k = 1$ and $m = -1$, then (1.1) will be the joint *pdf* of the first n upper record values of the $i.i.d.$ (independent and identically distributed) $rv\,'s$ with *cdf* F and *pdf* f.

Integrating out $x_1, x_2, ..., x_{r-1}, x_{r+1}, ..., x_n$ from (1.1) we obtain the *pdf* $f_{r,n,m,k}$ of $X(r,n,m,k)$

$$f_{r,n,m,k}(x) = \frac{c_{r-1}}{(r-1)!}\left(\overline{F}(x)\right)^{\gamma_r - 1} f(x)\ g_m^{r-1}(F(x)), \tag{1.2}$$

where $c_{r-1} = \Pi_{j=1}^{r}\gamma_j$ and

$$\begin{aligned} g_m(x) &= h_m(x) - h_m(0) = \frac{1}{m+1}\left[1-(1-x)^{m+1}\right], \quad m \neq -1, \\ &= -ln(1-x), \quad m = -1, \qquad x \in (0,1) \end{aligned}$$

and

$$\begin{aligned} h_m(x) &= -\frac{1}{m+1}(1-x)^{m+1}, \quad m \neq -1, \\ &= -ln(1-x), \qquad m = -1. \qquad x \in (0,1) \end{aligned}$$

Note that, since $\lim_{m\to -1}\frac{1}{m+1}\left[1-(1-x)^{m+1}\right] = -ln(1-x)$, we will write

$$g_m(x) = \frac{1}{m+1}\left[1-(1-x)^{m+1}\right], \quad \textit{for all} \ x \in (0,1) \ \textit{and all} \ m \ \textit{with}$$
$$g_{-1}(x) = \lim_{m\to -1} g_m(x).$$

The joint *pdf* of $X(s,n,m,k)$ and $X(r,n,m,k)$, $r < s$, is given by (see Kamps [8], p.68)

$$\begin{aligned} f_{s,r,n,m,k}(x,y) &= \tfrac{c_{s-1}}{(r-1)!(s-r-1)!}\left(\overline{F}(x)\right)^{m-1} f(x)\, g_m^{r-1}(F(x)) \times \\ &[h_m(F(y)) - h_m(F(x))]^{s-r-1}\left(\overline{F}(y)\right)^{\gamma_s - 1} f(y), \quad y \geq x. \end{aligned}$$

Consequently, the conditional *pdf* of $X(s_1,n,m,k)$ given $X(r,n,m,k) = x$, for $s_1 = s + l$, $s > 2$ and $m \neq -1$, is

(1.3)

$$f_{s_1 \mid r,n,m,k}(y|x)$$
$$= \frac{c_{s_1-1}}{c_{r-1}(s_1-r-1)!}[h_m(F(y)) - h_m(F(x))]^{s_1-r-1}\frac{(\overline{F}(y))^{\gamma_{s_1}-1}}{(\overline{F}(x))^{\gamma_{r+1}}} f(y), \quad y > x.$$

Burkschat et al. [5] introduced lower generalized order statistics ($lgos$) as follows:

The $rv's$ $X^*(1,n,m,k), X^*(2,n,m,k), ..., X^*(n,n,m,k)$, $k>0$, $m\in\mathbb{R}$, are n $lgos$ from an absolutely continuous cdf F with corresponding pdf f if their joint pdf $f^*(x_1,x_2,...,x_n)$, can be written as

$$f^*(x_1,x_2,...,x_n)=k\left(\Pi_{j=1}^{n-1}\gamma_j\right)\left[\Pi_{j=1}^{n-1}(F(x_j))^m\ f(x_j)\right]\times \tag{1.4}$$
$$(F(x_n))^{k-1}f(x_n),\ F^{-1}(1-)>x_1>x_2>...>x_n>F^{-1}(0+).$$

The marginal pdf $f^*_{r,n,m,k}(x)$ is

$$f^*_{r,n,m,k}(x)=\frac{c_{r-1}}{(r-1)!}(F(x))^{\gamma_r-1}f(x)\,q_m^{r-1}(F(x)), \tag{1.5}$$

where

$$q_m=\frac{1}{m+1}(1-x)^{m+1},\quad for\quad m\neq-1$$
$$=-ln(x),\quad for\quad m=-1.$$

The joint pdf of $X^*(s,n,m,k)$ and $X^*(r,n,m,k)$, $r<s$, is given by

(1.6)

$$f_{s,r,n,m,k}(x,y)=\frac{c_{s-1}}{(r-1)!(s-r-1)!}(F(x))^{m-1}f(x)\,v_m^{r-1}(F(x))\times$$
$$[h_m^*(F(y))-h_m^*(F(x))]^{s-r-1}\left(\overline{F}(y)\right)^{\gamma_s-1}f(y),x\geq y,$$

where

$$h_m^*(x)=-\frac{1}{m+1}x^{m+1},\quad for\quad m\neq-1$$
$$=-ln(x),\quad for\quad m=-1.$$

Consequently, the conditional pdf of $X^*(s_1,n,m,k)$ given $X^*(r,n,m,k=x$, is

(1.7)

$$f^*_{s_1\mid r,n,m,k}(y|x)=\frac{c_{s_1-1}}{c_{r-1}(s_1-r-1)!}[h_m^*(F(y))-h_m^*(F(x))]^{s_1-r-1}\times$$
$$\frac{(F(y))^{\gamma_{s_1}-1}}{(F(x))^{\gamma_{r+1}}}f(y),\ y<x.$$

2. Characterization results

Characterizations of probability distributions by different regression conditions on generalized order statistics has attracted the attention of many researchers (e.g., Bieniek and Szynal [4], Cramer et al. [6] and Bieniek [3]), to name a few. In [4] , the authors consider all *cdf 's* F for which the following linearity of regression holds:

$$E\left[\,(X\ (r+l,n,m,k))\mid X\,(r,n,m,k)\right]=a\ X\,(r,n,m,k)+b.$$

They conclude that only exponential, Pareto and power function distributions satisfy this equation. Using this result they obtain characterizations of these distributions based on sequential order statistics, records and progressive type II censored order statistics. Cramer et al. [6] , point out that characterizations of distributions based on linear regressions

$$E\left[\psi\ (X\ (r+l,n,m,k))\mid X\,(r,n,m,k)=\cdot\right]=\varphi\,(\cdot) \tag{2.1}$$

have been studied extensively for order statistics and record values ($r\in\mathbb{N}$, $l=1$. Since *gos* provide a unifying approach to these models, they set up a comprehensive solution related to characterization problems. They started with the case of adjacent *gos* $(l=1)$ and then pointed out that for larger l the calculations become more difficult. In order to obtain an explicit result, they restricted themselves to a linear function φ . They showed that the linearity of the conditional expectation provides a characterization of generalized Pareto distributions. Ahsanullah and Hamedani [1] presented characterizations of continuous distributions based on (2.1) for $l=1$ but without the assumptions of monotonicity of $\psi\,(\cdot)$ and linearity of $\varphi\,(\cdot)$. Jin and Lee [7] presented the characterizations of Pareto and Weibull distributions based on conditional expectations of the upper record values. Following [7] we present similar characterization based on conditional expectations of *gos* extending the characterization results of Jin and Lee [7]. We also present a characterization of the power function distribution based on the conditional expectation of lower generalized order statistics.

Theorem 2.1. *Let* $\{X_n\ ,\ n\geq 1\}$ *be a sequence of i.i.d. random variables with an absolutely continuous cdf* $F\,(x)$ *, corresponding pdf* $f\,(x)$ *and* $E\,[X^p]<\infty$, $p\in\mathbb{N}$. *For* $l\in\mathbb{N}$ *and* $p<\theta\in\mathbb{R}^+$

(2.2)

$$\frac{(\theta\gamma_{s+1}-p)(\theta\gamma_{s+2}-p)....(\theta\gamma_{s+l}-p)}{\gamma_{s+1}\gamma_{s+2}....\gamma_{s+l}}E(X^p(s+l,n,m,k)\mid X(r,n,m,k)=x)$$

$$= \theta^l E(X^p(s+l,n,m,k) \mid X(r,n,m,k)=x),$$

if and only if

$$F(x) = 1 - x^{-\theta}, \quad x \geq 1. \tag{2.3}$$

Proof. Suppose (2.3) holds and as before, let $s_1 = s+l$. Then

$$f_{s_1 \mid r,n,m,k}(y|x) = \frac{c_{s_1-1}}{c_{r-1}(s_1-r-1)!}(\frac{1}{m+1})^{s_1-r-1} \times$$
$$(-\frac{1}{y^{\theta(m+1)}} + \frac{1}{x^{\theta(m+1)}})^{s_1-r-1}(\frac{1}{y^\theta})^{\gamma_{s_1}-1}(x^\theta)^{\gamma_{r+1}}\frac{\theta}{y^{\theta+1}},$$

and

$$E(X^p(s_1,n,m,k) \mid X(r,n,m,k)=x) = \frac{c_{s_1-1}\theta(x^\theta)^{\gamma_{r+1}}}{c_{r-1}(s_1-r-1)!}(\frac{1}{m+1})^{s_1-r-1} \times$$
$$\int_x^\infty y^{-(\theta\gamma_{s_1}-p)-1}(-\frac{1}{y^{\theta(m+1)}} + \frac{1}{x^{\theta(m+1)}})^{s_1-r-1}dy.$$

Since $\theta - p > 0$, $\gamma_{s_1} > 1$, then $\theta\gamma_{s_1} - p > 0$ and using integration by parts on the right hand side of the above equality, we arrive at

$$E(X^p(s_1,n,m,k) \mid X(r,n,m,k)=x)$$
$$= \frac{c_{s_1-1}\theta^2(x^\theta)^{\gamma_{r+1}}}{c_{r-1}(s_1-r-2)!}\frac{1}{(\theta\gamma_{s_1}-p)}(\frac{1}{m+1})^{s_1-r-2} \times$$
$$\int_x^\infty y^{-(\theta\gamma_{s_1}-p)-\theta(m+1)-1}(-\frac{1}{y^{\theta(m+1)}} + \frac{1}{x^{\theta(m+1)}})^{s_1-r-2}dy.$$

Observing that $(\theta\gamma_{s_1}-p) + \theta(m+1) = \theta\gamma_{s_1-1} - p$, we have

$$E(X^p(s_1,n,m,k) \mid X(r,n,m,k)=x)$$
$$= \frac{c_{s_1-1}\theta^2(x^\theta)^{\gamma_{r+1}}}{c_{r-1}(s_1-r-2)!}\frac{1}{(\theta\gamma_{s_1}-p)}(\frac{1}{m+1})^{s_1-r-2} \times$$
$$\int_x^\infty y^{-(\theta\gamma_{s_1-1}-p)-1}(-\frac{1}{y^{\theta(m+1)}} + \frac{1}{x^{\theta(m+1)}})^{s_1-r-2}dy.$$

Employing integration by parts on the right hand side of the above equality, results in

$$E(X^p(s_1,n,m,k) \mid X(r,n,m,k)=x)$$
$$= \frac{c_{s_1-1}\theta^3(x^\theta)^{\gamma_{r+1}}}{c_{r-1}(s_1-r-3)!}\frac{1}{(\theta\gamma_{s_1}-p)(\theta\gamma_{s_1-1}-p)}(\frac{1}{m+1})^{s_1-r-3} \times$$
$$\int_x^\infty y^{-(\theta\gamma_{s_1-1}-p)-\theta(m+1)-1}(-\frac{1}{y^{\theta(m+1)}} + \frac{1}{x^{\theta(m+1)}})^{s_1-r-3}dy.$$

Continuing in this manner, we arrive at

$$
\begin{aligned}
&E(X^p(s_1,n,m,k) \mid X(r,n,m,k)=x)\\
&= \frac{c_{s_1-1}\theta^l (x^\theta)^{\gamma_{r+1}}}{c_{r-1}(s_1-r-l)!} \frac{1}{\Pi_{i=1}^{l}\left(\theta\gamma_{s_1-i}-p\right)} (\frac{1}{m+1})^{s_1-r-l} \times \\
&\quad \int_x^\infty y^{-\left(\theta\gamma_{s_1--l}-p\right)-1} (-\frac{1}{y^{\theta(m+1)}} + \frac{1}{x^{\theta(m+1)}})^{s_1-r-l} dy.\\
&= \frac{\theta^l \Pi_{i=s+1}^{s+l}\gamma_i}{\Pi_{i=1}^{l}\left(\theta\gamma_{s_1-i}-p\right)} E(X^p(s,n,m,k) \mid X(r,n,m,k)=x),
\end{aligned}
$$

which is now (2.2).

Now assume (2.2) holds. Then

(2.4)

$$
\begin{aligned}
&\frac{(\theta\gamma_{s+1}-p)(\theta\gamma_{s+2}-p)....(\theta\gamma_{s+l}-p)}{\gamma_{s+1}\gamma_{s+2}....\gamma_{s+l}(s_1-r-1)!} \times \\
&\quad \int_x^\infty y^p [h_m(F(y)) - h_m(F(x))]^{s_1-r-1} (1-F(y))^{\gamma_{s_1}-1} f(y) dy\\
&= \frac{\theta^l}{(s-r-1)!} \int_x^\infty y^p [h_m(F(y)) - h_m(F(x))]^{s-r-1} (1-F(y))^{\gamma_s-1} f(y) dy.
\end{aligned}
$$

Differentiating both sides of (2.4) $(s-r)$ times with respect to x, we obtain

$$
\begin{aligned}
&\int_x^\infty y^p [h_m(F(y)) - h_m(F(x))]^{l-1} (1-F(y))^{\gamma_{s_1}-1} f(y) dy\\
&= \frac{\theta^l \gamma_{s+1}\gamma_{s+2}....\gamma_{s+l}}{(\theta\gamma_{s+1}-p)(\theta\gamma_{s+2}-p)....(\theta\gamma_{s_1}-p)} x^p \left(1-F(x)\right)^{\gamma_s+1}.
\end{aligned}
\tag{2.5}
$$

Letting $u = \left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s}$ in (2.5), we arrive at

$$
\begin{aligned}
&\int_0^1 \left(\overline{F}^{-1}\left(u^{1/\gamma_s}\overline{F}(x)\right)\right)^p \left[\frac{\left(\overline{F}(x)\right)^{m+1} - \left(\overline{F}(x)\, u^{1/\gamma_s}\right)^{m+1}}{m+1}\right]^{l-1} \times \\
&\quad \left(u^{1/\gamma_s}\overline{F}(x)\right)^{\gamma_{s_1}-1} \left(\frac{1}{\gamma_s} u^{\frac{1-\gamma_s}{\gamma_s}} \overline{F}(x)\right) du\\
&= \frac{\theta^l \gamma_{s+1}\gamma_{s+2}....\gamma_{s+l}}{(\theta\gamma_{s+1}-p)(\theta\gamma_{s+2}-p)....(\theta\gamma_{s_1}-p)} x^p \left(1-F(x)\right)^{\gamma_s+1},
\end{aligned}
$$

or

$$\frac{(\theta\gamma_{s+1}-p)(\theta\gamma_{s+2}-p)....(\theta\gamma_{s_1}-p)}{\gamma_{s+1}\gamma_{s+2}....\gamma_{s+l}}\times$$
$$\int_0^1 \left(\overline{F}^{-1}\left(u^{1/\gamma_s}\overline{F}(x)\right)\right)^p \left(\frac{1-u^{\frac{m+1}{\gamma_s}}}{m+1}\right)^{l-1} du = \theta^l x^p,$$

and letting $v = u^{1/\gamma_s}$ and $\overline{F}(x) = t$, we have

$$\frac{(\theta\gamma_{s+1}-p)(\theta\gamma_{s+2}-p)....(\theta\gamma_{s_1}-p)}{\gamma_{s+1}\gamma_{s+2}....\gamma_{s+l}}\times$$
$$\int_0^1 \left(\overline{F}^{-1}(vt)\right)^p \left(\frac{1-v^{m+1}}{m+1}\right)^{l-1} v^{\gamma_s-1}dv = \theta^l \left(\overline{F}^{-1}(t)\right)^p . \tag{2.6}$$

Putting $v = e^{-u}$ and $t = e^{-w}$ in (2.6) , we obtain, upon simplification

$$\frac{(\theta\gamma_{s+1}-p)(\theta\gamma_{s+2}-p)....(\theta\gamma_{s_1}-p)}{\gamma_{s+1}\gamma_{s+2}....\gamma_{s+l}}\times$$
$$\int_0^\infty \left(\overline{F}^{-1}\left(e^{-(u+w)}\right)\right)^p \left(\frac{1-e^{(m+1)u}}{m+1}\right)^{l-1} e^{-\gamma_s u}du = \theta^l \left(\overline{F}^{-1}\left(e^{-w}\right)\right)^p,$$

for $0 < w < \infty$.

Now, the rest follows from the proof of Theorem 2.1 of Jin and Lee ([7], page 245). □

We need the following definition for our characterization of the Weibull distribution.

Definition 2.2. The random variable X with *cdf* F belongs to the class C if for some $\delta > 0$, either $(\overline{F}(x+y))^{1/\delta} \geq (\overline{F}(x))^{1/\delta}(\overline{F}(y))^{1/\delta}$, for all $x, y \geq 0$, or $(\overline{F}(x+y))^{1/\delta} \leq (\overline{F}(x))^{1/\delta}(\overline{F}(y))^{1/\delta}$, for all $x, y \geq 0$.

Theorem 2.3. *Let $\{X_n\ ,\ n \geq 1\}$ be a sequence of i.i.d. random variables with an absolutely continuous cdf $F(x)$, corresponding pdf $f(x)$ and $E\left[X^\delta\right] < \infty$, $\delta > 0$. We assume that $F(0) = 0$ and $F(x) > 0$ for all $x > 0$. For $l \in \mathbb{N}$, the following statements are equivalent:*

(*i*) $F(x) = 1 - e^{-x^\delta}$, $x \geq 0$.

(ii) *The random variable* X *belongs to the class* C *and*

(2.7)
$$E\left[(X(s+l,m.n.k))^{\delta} \mid (X(r,m.n.k)=x\right]$$
$$=\frac{1}{\gamma_{s+l}}+\frac{1}{\gamma_{s+l-1}}+...+\frac{1}{\gamma_{s+1}}+E\left[(X(s,m.n.k))^{\delta} \mid (X(r,m.n.k)=x\right].$$

Proof. Suppose (i) holds and let $Y=X^{\delta}$, then Y has an exponential distribution with parameter 1. We know that (see, [2], page 71)

$$Y(r,n,m,k)\stackrel{d}{=}\sum_{j=1}^{r}\frac{Y_j}{\gamma_j},$$
$$Y(s,n,m,k)\stackrel{d}{=}\sum_{j=1}^{s}\frac{Y_j}{\gamma_j} \quad and \quad Y(s+l,n,m,k)\stackrel{d}{=}\sum_{j=1}^{s+l}\frac{Y_j}{\gamma_j},$$

where $Y_j, j=1,2,...,s+l$ *are i.i.d with cdf* $F_Y(y)=1-e^{-y}, y\geq 0$, and

$$E(Y(s,n,m,k))=\sum_{j=1}^{s}\frac{1}{\gamma_j} \quad and \quad E(Y(s+l,n,m,k))=\sum_{j=1}^{s+l}\frac{1}{\gamma_j}.$$

Thus,

$$E(Y(s+l,n,m,k)|E(r,n,m,k)=t)$$
$$=\sum_{j=s+1}^{s+l}\frac{1}{\gamma_j}+E(Y(s,n,m,k)|Y(r,n,m,k)=t),\ s>r.$$

Writing the above equation in terms of X, we have (2.7).

Now, assume (ii) holds. Then, the left hand side of (2.7) can be expressed as

$$E\left[(X(s+l,m.n.k))^{\delta} \mid (X(r,m.n.k)=x\right]$$
$$=\frac{c_{s+l-1}}{c_{r-1}}\frac{1}{(s+l-r-1)!}\int_x^{\infty}y^{\delta}(h_m(y)-h_m(x))^{s+l-r-1}\frac{(\overline{F}(y))^{\gamma_{s+l}-1}}{(\overline{F}(x))^{\gamma_{r+1}}}f(y)dy.$$

Thus,

(2.8)
$$\frac{c_{s+l-1}}{c_{r-1}} \frac{1}{(s+l-r-1)!} \int_x^{\infty} y^{\delta} (h_m(y) - h_m(x))^{s+l-r-1} (\overline{F}(y))^{\gamma_{s+l}-1} f(y) dy$$
$$= \left(\frac{1}{\gamma_{s+l}} + \frac{1}{\gamma_{s+l-1}} + ... + \frac{1}{\gamma_{s+1}} \right) (\overline{F}(x))^{\gamma_{r+1}}$$
$$+ \frac{c_{s-1}}{c_{r-1}} \frac{1}{(s-r-1)!} \int_x^{\infty} y^{\delta} (h_m(y) - h_m(x))^{s-r-1} (\overline{F}(y))^{\gamma_s - 1} f(y) dy.$$

Now, differentiating both sides of (2.8) with respect to x, $(s-r)$ times, we arrive at

$$\frac{c_{s+l-1}}{c_{r-1}} \frac{1}{(l-1)!} \int_x^{\infty} y^{\delta} (h_m(y) - h_m(x))^{l-1} (\overline{F}(y))^{\gamma_{s+l}-1} f(y) dy$$
$$= \gamma_{r+1}\gamma_{r+2}...\gamma_s (\frac{1}{\gamma_{s+l}} + \frac{1}{\gamma_{s+l-1}} + ... + \frac{1}{\gamma_{s+1}}) (\overline{F}(x))^{\gamma_{r+1}} + \frac{c_{s-1}}{c_{r-1}} x^{\delta} (\overline{F}(x))^{\gamma_{r+1}}.$$

We can write the above equation as

$$\begin{aligned} &\frac{1}{(l-1)!} \int_x^{\infty} y^{\delta} \left(\frac{h_m(y) - h_m(x)}{\overline{F}(x)} \right)^{l-1} \left(\frac{\overline{F}(y)}{\overline{F}(x)} \right)^{\gamma_{s+l}-1} \frac{f(y)}{\overline{F}(x)} dy \\ &= \frac{c_{s-1}}{c_{s+l-1}} (\frac{1}{\gamma_{s+l}} + \frac{1}{\gamma_{s+l-1}} + ... + \frac{1}{\gamma_{s+1}}) + \frac{c_{s-1}}{c_{s+l-1}} x^{\delta}. \end{aligned} \tag{2.9}$$

Putting $u = \frac{\overline{F}(y)}{\overline{F}(x)}$ in (2.9), we obtain

$$\begin{aligned} &\frac{1}{(l-1)!} \int_0^1 \left(\overline{F}^{-1}(u\overline{F}(x)) \right)^{\delta} (1 - u^{m+1})^{l-1} u^{\gamma_{s+l}-1} du \\ &= \frac{c_{s-1}}{c_{s+l-1}} (\frac{1}{\gamma_{s+l}} + \frac{1}{\gamma_{s+l-1}} + ... + \frac{1}{\gamma_{s+1}}) + \frac{c_{s-1}}{c_{s+l-1}} x^{\delta}. \end{aligned} \tag{2.10}$$

Substituting $u = e^{-v}$ and $\overline{F}(x) = e^{-w}$ in (2.10), we have

(2.11)
$$\frac{1}{(l-1)!} \int_0^{\infty} \left(\overline{F}^{-1}(e^{-(w+v)}) \right)^{\delta} (1 - e^{-(m+1)v})^{l-1} (e^{-v})^{\gamma_{s+l}} dv$$
$$- \frac{c_{s-1}}{c_{s+l-1}} \left(\overline{F}^{-1}(e^{-w}) \right)^{\delta} = \frac{c_{s-1}}{c_{s+l-1}} (\frac{1}{\gamma_{s+l}} + \frac{1}{\gamma_{s+l-1}} + ... + \frac{1}{\gamma_{s+1}}).$$

We observe that the right hand side of (2.11) is independent of w , so letting $w = 0$ in (2.11) and noting that $\overline{F}^{-1}(1) = 0$, we arrive at

$$\begin{aligned}&\frac{1}{(l-1)!}\int_0^\infty \left(\overline{F}^{-1}(e^{-v})\right)^\delta (1-e^{-(m+1)v})^{l-1}(e^{-v})^{\gamma_{s+l}}dv \\ &= \frac{c_{s-1}}{c_{s+l-1}}(\frac{1}{\gamma_{s+l}} + \frac{1}{\gamma_{s+l-1}} + ... + \frac{1}{\gamma_{s+1}}).\end{aligned} \tag{2.12}$$

Now, in view of (2.11) and (2.12), we have

$$\begin{aligned}&\frac{1}{(l-1)!}\int_0^\infty \left(\overline{F}^{-1}(e^{-(w+v)})\right)^\delta (1-e^{-(m+1)v})^{l-1}(e^{-v})^{\gamma_{s+l}}dv \\ &\quad - \frac{1}{(l-1)!}\int_0^\infty \left(\overline{F}^{-1}(e^{-v})\right)^\delta (1-e^{-(m+1)v})^{l-1}(e^{-v})^{\gamma_{s+l}}dv \\ &= \frac{c_{s-1}}{c_{s+l-1}}\left(\overline{F}^{-1}(e^{-w})\right)^\delta .\end{aligned} \tag{2.13}$$

Let $H = \int_0^\infty (1-e^{-(m+1)v})^{l-1}(e^{-v})^{\gamma_{s+l}}dv$. Putting $z = (m+1)v$, we have

$$\begin{aligned}H &= \int_0^\infty (1-e^{-(m+1)v})^{l-1}(e^{-v})^{\gamma_{s+l}}dv \\ &= \frac{1}{m+1}\int_0^\infty (1-e^{-z})^{l-1}e^{-\frac{\gamma_{s+l}}{m+1}z}dz.\end{aligned} \tag{2.14}$$

Upon integration by parts on the right hand side of (2.14) , l time and substituting $v = \frac{(m+1)}{z}$, we have

$$\begin{aligned}&\frac{1}{(l-1)!}\int_0^\infty \left(\overline{F}^{-1}(e^{-v})\right)^\delta (1-e^{-(m+1)v})^{l-1}(e^{-v})^{\gamma_{s+l}}dv \\ &= \frac{c_{s-1}}{c_{s+l-1}}\left(\overline{F}^{-1}(e^{-w})\right)^\delta .\end{aligned} \tag{2.15}$$

In view of (2.13) and (2.15) , we obtain

$$\begin{aligned}&\int_0^\infty \left(\overline{F}^{-1}(e^{-(w+v)})\right)^\delta (1-e^{-(m+1)v})^{l-1}(e^{-v})^{\gamma_{s+l}}dv \\ &\quad - \int_0^\infty \left(\overline{F}^{-1}(e^{-v})\right)^\delta (1-e^{-(m+1)v})^{l-1}(e^{-v})^{\gamma_{s+l}}dv \\ &= \int_0^\infty \left(\overline{F}^{-1}(e^{-w})\right)^\delta (1-e^{-(m+1)v})^{l-1}(e^{-v})^{\gamma_{s+l}}dv.\end{aligned}$$

Upon simplification of the above equality, we arrive at

$$\begin{aligned}\int_0^\infty &\left[\left(\overline{F}^{-1}(e^{-(w+v)})\right)^\delta - \left(\overline{F}^{-1}(e^{-v})\right)^\delta - \left(\overline{F}^{-1}(e^{-w})\right)^\delta\right] \times \\ &(1-e^{-(m+1)v})^{l-1}(e^{-v})^{\gamma_{s+l}}dv \\ &= 0.\end{aligned} \tag{2.16}$$

In view of the fact that X belongs to the class C , we must have

$$\left(\overline{F}^{-1}(e^{-(w+v)})\right)^\delta = \left(\overline{F}^{-1}(e^{-v})\right)^\delta + \left(\overline{F}^{-1}(e^{-w})\right)^\delta , \; for\ all \;\; v\ and\ w \geq 0. \tag{2.17}$$

Putting $G(u) = \left(\overline{F}^{-1}(e^{-u})\right)^\delta$, we can write (2.17) as

$$G(x+y) = G(x) + G(y) \;\; , \; for\ all \;\; x\ and\ y \geq 0. \tag{2.18}$$

Equation (2.18) is the well-known Cauchy functional equation whose solution is $G(x) = \frac{x}{\theta}$, for all $x \geq 0$ where θ is a constant. Thus $\left(\overline{F}^{-1}(e^{-x})\right)^\delta = \frac{x}{\theta}$ from which we have $F(x) = 1 - e^{-\theta x}$. In view of the boundary conditions $F(0) = 0$ and $\lim_{x\to\infty} F(x) = 1$, we must have

$$F(x) = 1 - e^{-\theta x^\delta}, \; x \geq 0, \; \theta > 0, \; \delta > 0. \tag{2.19}$$

□

Note that we can assume without loss of generality $\theta = 1$.

Remark 2.4. For $\delta = 1$, Theorem 2.2 gives a characterization of the exponential distribution based on the generalized order statistics.

The following theorem gives characterization of power function distribution using *lgos*.

Theorem 2.5. *Let* $\{X_n \; , \; n \geq 1\}$ *be a sequence of i.i.d. random variables with an absolutely continuous cdf* $F(x)$ *, corresponding pdf* $f(x)$ *and* $E[X^p] < \infty$. *For* $\theta > 0$ *and* $l \in \mathbb{N}$

$$\begin{aligned}&\frac{(\theta\gamma_{s+1}+p)(\theta\gamma_{s+2}+p)....(\theta\gamma_{s+l}+p)}{\gamma_{s+1}\gamma_{s+2}....\gamma_{s+l}} \times \\ &E(((X^*(s+l,n,m,k))^p \; |(X(r,n,m,k)=x) \\ &\quad = \theta^l E((X^*(s,n,m,k))^p \mid X(r,n,m,k)=x),\end{aligned} \tag{2.20}$$

if and only if

$$F(x) = x^\theta, \quad 0 \leq x \leq 1. \tag{2.21}$$

Proof. Is similar to that of Theorem 2.1. □

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Department of Management Sciences
Rider University
Lawrenceville, NJ 08648-3009
E-mail: ahsan@rider.edu

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Department of Mathematics, Statistics and Computer Science
Marquette University
Milwaukee, WI 53201-1881
E-mail: g.hamedani@mu.edu